JOURNAL OF APPROXIMATION THEORY 28, 1–19 (1980)

Approximations of Contraction Operators on L^∞, II

CHOO-WHAN KIM*

Department of Mathematics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6

Communicated by Oved Shisha

Received December 12, 1977

1. INTRODUCTION

In a preceding paper [12] we have proved approximation theorems for contraction operators on $L^\infty[0, 1]$. The purpose of the present paper is to prove analogues of the results of [12] for contraction operators on $L^\infty[0, \infty)$. We use the nonnegative real half-line for simplicity of notation; our results hold also when the underlying space is the real line.

Let $(X, \mathscr{F}, \mu)$ denote the measure space consisting of the nonnegative real half-line, the Lebesgue measureable sets and Lebesgue measure. For brevity, the measure space $(X, \mathscr{F}, \mu)$ will be denoted by (X, μ). On X, we shall consider only $\mathscr{F}$-measurable real functions (modulo μ-equivalence), and by a set on X we shall always mean an element of $\mathscr{F}$. Relations among sets and functions are understood to hold modulo sets of μ-measure zero.

Assume $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. We shall denote $L^p(X, \mathscr{F}, \mu)$ by $L^p(X, \mu)$, or, briefly by L^p. Let $[L^p, L^q]$ be the set of real bounded operators from L^p into L^q, and let $[L^p] = [L^p, L^p]$. It is well-known that $[L^q, L^q]$ is an order complete vector lattice. For each $T \in [L^p]$, the positive operators $T^+ = T \vee 0$, $T^- = (-T) \vee 0$ and $|T| = T^+ + T^-$ are called the positive part, the negative part, and the linear modulus of T, respectively (see [3, 2, 16] for details).

Let $\mathscr{C}_p$ denote the set of contraction operators on L^p, that is, $T \in [L^p]$ and $\|T\|_p \leqslant 1$. Let $\mathscr{C}_1^*$ denote the set of adjoints of operators in $\mathscr{C}_1$. Define the sets $\mathscr{C}$, $\mathscr{M}$, $\mathscr{D}$ and $\mathfrak{D}$ as follows:

$$\mathscr{C} = \mathscr{C}_\infty, \qquad \mathscr{M} = \mathscr{C}_1^*, \qquad \mathscr{D} = \mathscr{C}_1 \cap \mathscr{C}_1^*, \qquad \mathfrak{D} = \mathscr{C}_1 \cap \mathscr{C}_\infty .$$

It is easily seen that $\mathscr{D} \subsetneqq \mathscr{M} \subsetneqq \mathscr{C}$ and $\mathfrak{D} \subsetneqq \mathscr{C}$. By the Riesz convexity theorem

* Work on this paper was supported by the National Research Council of Canada, Grant A4844.

0021-9045/80/010001-19$02.00/0

[3, p. 525] we have $\mathfrak{D} = \bigcap_{1 \leqslant p \leqslant \infty} \mathscr{C}_p$. When the underlying space is a finite measure space, we have $\mathscr{D} = \mathfrak{D}$. However, at the present case, an example of Brown [1, p. 370] shows that $\mathscr{D} \subsetneqq \mathfrak{D}$. Note that for each T in $[L^1]$, $T^{*+} = T^{+*}$, $T^{*-} = T^{-*}$, and $| T^* | = | T |^*$ [12, Lemma 1]. It follows readily that $\mathscr{C}, \mathscr{M}, \mathscr{D}$ and $\mathfrak{D}$ are convex sublattices of the order complete Banach lattice $[L^\infty]$ and are semigroups under multiplication. In particular, $\mathscr{D}$ is self-adjoint, $\mathscr{D} = \mathscr{D}^*$, in the following sense. Each T in $\mathscr{D}$ as an element of $\mathscr{C}_1$ has the adjoint T^* in $\mathscr{C}$ satisfying the equation

$$\langle Tf, g \rangle = \langle f, T^*g \rangle \ (f \in L^1, g \in L^\infty).$$

We write $\langle f, g \rangle$ for $\int_X fg \, d\mu$ if the integral is well-defined. It follows from the above equation, together with T being an element of $\mathscr{D}$ that

$$|\langle f, T^*g \rangle| \leqslant \|f\|_\infty \|g\|_1 \wedge \|f\|_1 \|g\|_\infty \ (f, g \in L^1 \cap L^\infty).$$

We have then $\| T^*g \|_1 \leqslant \| g \|_1$ ($g \in L^1 \cap L^\infty$), so that T^* is uniquely extended to an element of $\mathscr{C}_1$, denoted also by T^*. Thus $T^* \in \mathscr{D}$. By repeating the above argument for $T^* \in \mathscr{D}$ we show that the adjoint $(T^*)^*$ of T^* is an element of $\mathscr{D}$, and $(T^*)^* = T$, so that $\mathscr{D} = \mathscr{D}^*$. On the other hand, it is easy to show that $\mathfrak{D}^* = \mathscr{D} \subsetneqq \mathfrak{D}$, so that $\mathfrak{D}$ is not self-adjoint.

Let $\mathscr{C}_0$ be the set of those T in $\mathscr{C}$ such that $| T | 1 = 1$. Define $\mathscr{M}_0$ and $\mathscr{D}_0$ by $\mathscr{M}_0 = \mathscr{M} \cap \mathscr{C}_0$ and $\mathscr{D}_0 = \{T \in \mathscr{D} : | T | 1 = 1, | T |^* 1 = 1\}$. For each $\mathscr{A} \subset \mathscr{C}$, let $\mathscr{A}^+$ denote the set of nonnegative elements of $\mathscr{A}$. An element of $\mathscr{M}^+$ [resp. $\mathscr{M}_0{}^+$] is called a sub-Markov [resp. Markov] operator, and an element of $\mathscr{D}^+$ [resp. $\mathscr{D}_0{}^+$] is called a doubly substochastic [resp. doubly stochastic] operator. Each homomorphism θ from the Boolean σ-algebra of measurable sets (modulo μ) into itself defines $T_\theta \in \mathscr{C}_0{}^+$ such that $T_\theta 1_A = 1_{\theta(A)}$. We write the indicator function of a set A by 1_A. Let $\mathscr{H}$ denote the set of such operators T_θ. For each nonsingular measurable point mapping φ from $(X, \mathscr{F}, \mu)$ into itself, that is $\mu(\varphi^{-1}(A)) = 0$ if $\mu(A) = 0$, we define $T_\varphi \in \mathscr{M}_0{}^+$ by $T_\varphi f(x) = f(\varphi(x))$. The set of such opeators T_φ is denoted by Ψ. Let Ψ_1 be the set of operators $T_\varphi \in \Psi$ for which φ is an injection. A measurable point mapping $\varphi : X \to X$ is called measure preserving if $\mu(\varphi^{-1}(A)) = \mu(A)$, where $A \subset \varphi(X)$, and a measure-preserving surjection $\varphi : X \to X$ is called invertible if φ is a bijection, and φ^{-1} is measurable. Clearly each measure-preserving map is nonsingular. Let Φ be the set of $T_\varphi \in \Psi$ for which φ is a measure preserving, and let Φ_1 be the set of $T_\varphi \in \Phi$ for which φ is invertible. Note that $\Phi \subset \mathscr{D}^+$ and $\Phi_1 \subset \mathscr{D}_0{}^+$.

For each $E \subset X$, define the operator I_E in $\mathscr{D}^+$ by $I_E f(x) = 1_E(x) f(x)$. Let $\sum$ be the family of all ordered pairs $\sigma = (A, B)$ of disjoint sets such that $X = A \cup B$, $\mu(A) \geqslant 0$, and $\mu(B) \geqslant 0$. Each $\sigma \in \sum$ defines the operator I_σ in

$\mathscr{D}_0$ by $I_\sigma = I_A - I_B$. Note that $|I_\sigma| = I$, where I denotes the identity operator. We also write $\sum$ for $\{I_\sigma : \sigma \in \sum\}$. Define

$$\tilde{\mathscr{H}} = \sum \mathscr{H}, \quad \tilde{\Psi} = \sum \Psi, \quad \tilde{\Psi}_1 = \sum \Psi_1, \quad \tilde{\Phi} = \sum \Phi, \quad \tilde{\Phi}_1 = \sum \Phi_1 .$$

We also define $\Psi' = \{I_E T : E \subset X, T \in \Psi\}$ and $\tilde{\Psi}' = \{I_E T : E \subset X, T \in \tilde{\Psi}\}$. Similarly the sets Ψ_1', $\tilde{\Phi}'$ and $\tilde{\Phi}_1'$ are defined.

For $\mathscr{A} \subset \mathscr{C}$, let ch $\mathscr{A}$ denote the convex hull of $\mathscr{A}$, and let ext $\mathscr{A}$ denote the set of extreme points of $\mathscr{A}$. By a minor modification of the proof of Theorem 1 in [11, p. 103] (see, also [12, Proposition 1]) we obtain

$$\operatorname{ext} \mathscr{C} = \tilde{\mathscr{H}} \text{ and } \operatorname{ext} \mathscr{M} = \mathscr{M} \cap \operatorname{ext} \mathscr{C} = \tilde{\Psi}.$$

On the other hand, we show that

$$\tilde{\Phi} \subsetneqq \mathscr{D} \cap \operatorname{ext} \mathscr{M} \subsetneqq \operatorname{ext} \mathscr{D}.$$

Since $\operatorname{ext} \mathscr{M} = \tilde{\Psi}$, it is easily seen that $\tilde{\Phi} \subset \mathscr{D} \cap \tilde{\Psi} \subset \operatorname{ext} \mathscr{D}$. Define the operatots T_φ, S_1 and S_2 as follows:

$$T_\varphi f(x) = f(4x)\, 1_{[0,\frac{1}{4})}(x) + f(4x - 2)\, 1_{[\frac{1}{2},1)}(x) + f(x + 1)\, 1_{[1,\infty)}(x).$$

$$S_1 f(x) = \frac{1}{2} f\left(\frac{x}{4}\right) 1_{[0,2)}(x) + f(x - 1)\, 1_{[2,\infty)}(x),$$

$$S_2 f(x) = \frac{1}{2} f\left(\frac{x+2}{4}\right) 1_{[0,2)}(x) + f(x - 1)\, 1_{[2,\infty)}(x).$$

Let $S = (S_1 + S_2)/2$. We see readily that

$$T_\varphi \in \mathscr{D} \cap \tilde{\Psi} - \tilde{\Phi} \qquad \text{and} \qquad T_\varphi^* = S \in \operatorname{ext} \mathscr{D} - \mathscr{D} \cap \tilde{\Psi},$$

so that the desired conclusion follows. It is known [12, Proposition 3] that $\tilde{\Phi} = \mathscr{D} \cap \tilde{\Psi} \subsetneqq \operatorname{ext} \mathscr{D}$ when the underlying space is the unit interval.

In Section 2 we shall show that operators in $\mathscr{C}$ and $\mathscr{M}$ can be approximated by convex combinations of operators in $\tilde{\Psi}_1$ or $\tilde{\Psi}$ in the weak* operator and the strong operator topologies of $[L^\infty]$. The norm approximation theorem is also proved for Hilbert-Schmidt operators in $\mathscr{M}$.

It has been shown [1, 7, 14] that doubly substochastic operators $\mathscr{D}^+$ can be approximated by convex combinations of operators in Φ_1 in well-known operator topologies of $[L^p]$ ($p = 1$ or 2).

In Section 3 we shall determine the subset of $\mathscr{D}$ that are represented by certain signed measures defined on the product space $(X \times X, \mathscr{F} \times \mathscr{F})$. We also prove that $\mathscr{D}$ is the closure of $\tilde{\Phi}_1$ in the weak operator topology of $[L^2]$ and is the closed convex hull of $\tilde{\Phi}_1'$ in the strong* operator topology of $[L^2]$. The norm approximation theorem is also proved for Hilbert-Schmidt operators in $\mathscr{D}$.

2. Approximations of $\mathscr{C}$ and $\mathscr{M}$

Let (X_1, μ) denote the usual Lebesgue measure space on $X_1 = [0, 1]$, and let λ be a probability measure on $(X, \mathscr{F})$ that is equivalent to μ, $\lambda \equiv \mu$. In analogy with the sets $\mathscr{C}$, $\mathscr{C}_0$, $\mathscr{M}$, $\mathscr{M}_0$ and $\mathscr{D}$, we also define the subsets $\mathscr{C}(X, \lambda)$, $\mathscr{C}_0(X, \lambda)$, $\mathscr{M}(X, \lambda)$, $\mathscr{M}_0(X, \lambda)$, $\mathscr{D}(X, \lambda)$ of $[L^\infty(X, \lambda)]$ and the subsets $\mathscr{C}(X_1, \mu)$, $\mathscr{C}_0(X_1, \mu)$, $\mathscr{M}(X_1, \mu)$, $\mathscr{M}_0(X_1, \mu)$, $\mathscr{D}(X_1, \mu)$ of $[L^\infty(X_1, \mu)]$. Let $\mathscr{K}$ be the set of kernel operators T in $\mathscr{M}$, that is, $Tg(x) = \int_X t(x, y)\, g(y)\, d\mu(y)$ ($g \in L^\infty(X, \mu)$), where $\int_X |t(x, y)|\, d\mu(y) \leqslant 1$. Similarly we define the sets $\mathscr{K}(X, \lambda)$ and $\mathscr{K}(X_1, \mu)$. Note that the sets mentioned above are semigroups under multiplication. We shall show that the semigroups $\mathscr{C}$[resp. $\mathscr{M}$, $\mathscr{K}$] and $\mathscr{C}(X_1, \mu)$ [resp. $\mathscr{M}(X_1, \mu)$, $\mathscr{K}(X_1, \mu)$] are isomorphic, so that the approximation theorems of $\mathscr{C}(X_1, \mu)$ [resp. $\mathscr{M}(X_1, \mu)$, $\mathscr{K}(X_1, \mu)$] [12] give rise to those of $\mathscr{C}$ [resp. $\mathscr{M}$, $\mathscr{K}$].

It is well-known [4, p. 173; 15, p. 329] that (X_1, μ) and (X, λ) are isomorphic, that is, there exists a measurable bijections $\xi : X_1 \to X$ such that $\mu(\xi^{-1}(A)) = \lambda(A)$ $(A \subset X)$ and $\mu(B) = \lambda(\xi(B))$ $(B \subset X_1)$. Let T_ξ and T_η, $\eta = \xi^{-1}$, be the operators defined by

$$T_\xi f(x) = f(\xi(x)), f \in L^\infty(X, \lambda);\ T_\eta g(x) = g(\eta(x)), g \in L^\infty(X_1, \mu).$$

In the sequel, let

$$P = T_\xi \text{ and } P^* = T_\eta .$$

It is easily seen that, for $1 \leqslant p \leqslant \infty$, $P : L^p(X, \lambda) \to L^p(X_1, \mu)$ and $P^* : L^p(X_1, \mu) \to L^p(X, \lambda)$ are positive isometries and satisfy the equation

$$\int_{X_1} (Pf)\, g\, d\mu = \int_X f P^* g\, d\mu \qquad (f \in L^\infty(X, \lambda), g \in L^\infty(X, \mu)).$$

Lemma 1. *$\mathscr{C}$ and $\mathscr{C}(X_1, \mu)$ are isomorphic.*

Proof. Since $\mu \equiv \lambda$, we shall identify $L^\infty(X, \mu)$ and $\mathscr{C}$ with $L^\infty(X, \lambda)$ and $\mathscr{C}(X, \lambda)$, respectively. Thus it is enough to show that $\mathscr{C}(X, \lambda)$ and $\mathscr{C}(X_1, \mu)$ are isomorphic. It is straightforward to show that the map $h : \mathscr{C}(X, \lambda) \to \mathscr{C}(X_1, \mu)$ defined by $h(T) = PTP^*$ is an isomorphism of the semigroups $\mathscr{C}(X, \lambda)$ and $\mathscr{C}(X_1, \mu)$. If we write $\hat{T} = h(T)$, then $\hat{T} = PTP^*$, $T = P^*\hat{T}P$, and $\| T \|_\infty = \| \hat{T} \|_\infty$. This completes the proof.

Let $\mathscr{C}_1(X, \lambda)$ and $\mathscr{C}_1(X_1, \mu)$ be the sets of contraction operators on $L^1(X, \lambda)$ and $L^1(X_1, \mu)$, respectively. Let

$$u = d\lambda/d\mu \qquad \text{and} \quad v = d\mu/d\lambda.$$

Note that $0 < u \in L^1(X, \mu)$, $\int_X u\, d\mu = 1$, and $uv = 1$.

LEMMA 2. *$\mathcal{M}$ and $\mathcal{M}(X_1, \mu)$ are isomorphic.*

Proof. We show first that $\mathcal{C}_1$ and $\mathcal{C}_1(X, \lambda)$ are isomorphic. Define the operators M_u and M_v by

$$M_u f' = uf',\ f' \in L^1(X, \lambda);\ M_v f = vf,\ f \in L^1(X, \mu).$$

It follows that $M_u : L^1(X, \lambda) \to L^1(X, \mu)$ and $M_v : L^1(X, \mu) \to L^1(X, \lambda)$ are bijective, positive isometries such that $M_u M_v f = f$ and $M_v M_u f' = f'$, where $f \in L^1(X, \mu)$ and $f' \in L^1(X, \lambda)$. Note that the adjoints M_u^* and M_v^* are the identity operator on $L^\infty(X, \mu) = L^\infty(X, \lambda)$. We see readily that the map $k : \mathcal{C}_1(X, \mu) \to \mathcal{C}_1(X, \lambda)$ defined by $k(S) = M_v S M_u$ is an isomorphism. If we write $S' = k(S)$, then $S' = M_v S M_u$, $S = M_u S' M_v$, and $\| S \|_1 = \| S' \|_1$.

It follows from the above result that $\mathcal{M} = \mathcal{C}_1^*$ and $\mathcal{M}(X, \lambda) = \mathcal{C}_1^*(X, \lambda)$ are also isomorphic. Since $S'^* = M_u S^* M_v^* = S^*$, we have $\mathcal{M} = \mathcal{M}(X, \lambda)$.

We shall show that the isomorphism h between $\mathcal{C}(X, \lambda)$ and $\mathcal{C}(X_1, \mu)$ induces that of $\mathcal{M}(X, \lambda)$ and $\mathcal{M}(X_1, \mu)$. Extend the map h on $\mathcal{C}_1(X, \lambda)$ by $h(S') = PS'P^*$. We see at once that the map h is an isomorphism between $\mathcal{C}_1(X, \lambda)$ and $\mathcal{C}_1(X_1, \mu)$, so that the desired conclusion follows.

LEMMA 3. *$\mathcal{K}$ and $\mathcal{K}(X_1, \mu)$ are isomorphic.*

Proof. It is straightforward to show that for each $T \in \mathcal{M} = \mathcal{M}(X, \lambda)$, T as an element of $\mathcal{K}$, has the kernel $t(x, y)$ if and only if T, as an element of $\mathcal{K}(X, \lambda)$, has the kernel $t'(x, y) = t(x, y)\, v(y)$. Thus, $\mathcal{K} = \mathcal{K}(X, \lambda)$.

We shall show that the isomorphism h between $\mathcal{M}(X, \lambda)$ and $\mathcal{M}(X_1, \mu)$ induces an isomorphism between $\mathcal{K}(X, \lambda)$ and $\mathcal{K}(X_1, \mu)$. Note that the mapping $\zeta : (X_1 \times X_1, \mu \times \mu) \to (X \times X, \lambda \times \lambda)$ defined by $\zeta(x, y) = (\xi(x), \xi(y))$ is an invertible measure-preserving map with $\zeta^{-1}(x, y) = (\eta(x), \eta(y))$, $(x, y) \in X \times X$. Let $t'(x, y)$ be the kernel for $T \in \mathcal{K}(X, \lambda)$. It follows that $\hat{T} = PTP^*$ is an element of $\mathcal{M}(X_1, \mu)$, and

$$\hat{T}\hat{g}(x) = \int_{X_1} t'(\xi(x), \xi(y))\, \hat{g}(y)\, d\mu(y),$$

where $\hat{g} \in L^\infty(X_1, \mu)$ and $\int_{X_1} | t'(\xi(x), \xi(y))|\, d\mu(y) = \int_X | t'(\xi(x), z)|\, d\lambda(z) \leqslant 1$, so that $\hat{T} \in \mathcal{K}(X_1, \mu)$ has the kernel $\hat{t}(x, y) = t \circ \zeta(x, y)$. This completes the proof.

Let $\mathcal{K}_2$ denote the set of those $T \in \mathcal{K}$ such that its kernel is an element of $L^2(X \times X, \mu \times \mu)$. The elements of $\mathcal{K}_2$ are called Hilbert-Schmidt operators. Similarly we define the sets $\mathcal{K}_2(X, \lambda)$ and $\mathcal{K}_2(X_1, \mu)$. The following example shows that, in general, $\mathcal{K}_2 \neq \mathcal{K}_2(X, \lambda)$.

EXAMPLE 1. Define the functions u, v, and w by

$$u = \sum_{n=1}^{\infty} 2^{-n} 1_{X_n}, \qquad v = \sum_{n=1}^{\infty} 2^{n} 1_{X_n}, \qquad w = (\sqrt{2} - 1) \sum_{n=1}^{\infty} 2^{-n/2} 1_{X_n},$$

where $X_n = [n - 1, n)$ for $n \geqslant 1$. Let λ be a probability measure on X such that $u = d\lambda/d\mu$. If we define $t(x, y) = 1_{X_1}(x)\, w(y)$ and $t'(x, y) = t(x, y)\, v(y)$, where $(x, y) \in X \times X$, then

$$\int_X t(x, y)\, d\mu(y) = 1_{X_1}(x), \qquad \iint_{X\times X} t(x, y)^2\, d\mu(x)\, d\mu(y) = (\sqrt{2} - 1)^2,$$

$$\iint_{X\times X} t'(x, y)^2\, d\lambda(x)\, d\lambda(y) = \infty.$$

On the other hand, we may show readily that

$$\iint_{X\times X} |f(x, y)|^p\, d\lambda(x)\, d\lambda(y) = \iint_{X_1\times X_1} |f \circ \zeta(x, y)|^p\, d\mu(x)\, d\mu(y),$$

where $f \in L^p(X \times X, \lambda \times \lambda)$, $1 \leqslant p < \infty$, and that $\mathscr{K}_2(X, \lambda)$ and $\mathscr{K}_2(X_1, \mu)$ are isomorphic. Thus the isomorphism between $\mathscr{K}$ and $\mathscr{K}(X_1, \mu)$ does not induce the isomorphism between $\mathscr{K}_2$ and $\mathscr{K}_2(X_1, \mu)$.

We see readily that $\mathscr{D}(X, \lambda)$ and $\mathscr{D}(X_1, \mu)$ are isomorphic. However, the example below shows that the map h is not an isomorphism between $\mathscr{D}$ and $\mathscr{D}(X, \lambda)$.

EXAMPLE 2. Let $\varphi(x) = x + 1$ on X and $\psi(x) = x - 1$ on $Y = [1, \infty)$. If we define $T = T_\varphi$ and $S = I_Y T_\varphi$, then both T and S are elements of $\mathscr{D}$ such that $\| T \|_p = \| S \|_p = 1$ ($p = 1, \infty$) and $T = S^*$. Let u, v, and λ be as in Example 1. Define $S' = M_v S M_u$, $\hat{S} = PSP^*$ and $\hat{T} = PTP^*$. A simple calculation yields $S'1 = 21_Y$, so that $\int_X | Tg |\, d\lambda = \int_X | g |\, S'1 d\lambda = 2 \int_Y | g |\, d\lambda$, where $g \in L^\infty(X, \lambda)$. It follows that $\| T \|_{L^1(X,\lambda)} = 2$, so that $T \notin \mathscr{D}(X, \lambda)$ and $\hat{T} \notin \mathscr{D}(X_1, \mu)$. On the other hand, we show readily that $\| S \|_{L^1(X,\lambda)} = \frac{1}{2}$, so that $S \in \mathscr{D}(X, \lambda)$ and $\hat{S} \in \mathscr{D}(X_1, \mu)$.

Let $\Psi(X_1, \mu)$ be the set of operators T_φ in $\mathscr{M}^+(X_1, \mu)$ for which φ is a nonsingular measurable mapping from (X_1, μ) into itself. Let $\sum (X_1, \mu)$ be the family of all ordered pairs $\sigma = (A, B)$ of disjoint subsets of X_1 such that $X_1 = A \cup B$, $\mu(A) \geqslant 0$, and $\mu(B) \geqslant 0$. Let $\tilde{\Psi}(X_1, \mu)$ be the set of operators in $\mathscr{M}(X_1, \mu)$ of the form $(I_A - I_B)\, T$, where $T \in \Psi(X_1, \mu)$ and $(A, B) \in \sum (X_1, \mu)$. In analogy with the sets Ψ_1, $\tilde{\Psi}_1$, Ψ_1' and $\tilde{\Psi}_1'$ we also define the sets $\Psi_1(X_1, \mu)$, $\tilde{\Psi}_1(X_1, \mu)$, $\Psi_1'(X_1, \mu)$ and $\tilde{\Psi}'(X_1, \mu)$. Since $\mu = \lambda$ on X, we show readily that Ψ [resp. $\tilde{\Psi}$] and $\Psi(X_1, \mu)$ [resp. $\tilde{\Psi}(X_1, \mu)$] are isomorphic and $P\Psi P^* = \Psi(X_1, \mu)$ [resp. $P\tilde{\Psi}P^* = \tilde{\Psi}(X_1, \mu)$]. Similar remarks apply for Ψ_1 and $\Psi_1(X_1, \mu)$; $\tilde{\Psi}_1$ and $\tilde{\Psi}_1(X_1, \mu)$; Ψ_1' and $\Psi_1'(X_1, \mu)$.

THEOREM 1. *$\mathscr{C}$ [resp. $\mathscr{C}^+$, $\mathscr{C}_0^+$] is a compact convex set and is the closure of $\tilde{\Psi}_1$ [resp. Ψ_1', Ψ_1] in the weak* operator topology of* $[L^\infty(X, \mu)]$.

Proof. The convexity of $\mathscr{C}$ is clear, and the compactness of $\mathscr{C}$ follows from a weak-compactness principle of Kadison [5]. We shall show that $\tilde{\Psi}_1$ is dense in $\mathscr{C}$. Given T in $\mathscr{C}$, if we set $\hat{T} = PTP^*$, then $\hat{T}$ is in $\mathscr{C}(X_1, \mu)$ by Lemma 1. It follows from Theorem 1 of [12] that there exists a net $(\hat{T}_\alpha)_\alpha$ in $\tilde{\Psi}_1(X_1, \mu)$ that converges to $\hat{T}$ in the weak* operator topology of $[L^\infty(X_1, \mu)]$. Note that the net $(T_\alpha)_\alpha$ defined by $T_\alpha = P^*\hat{T}_\alpha P$ belongs to $\tilde{\Psi}_1$. For each $f \in L^1(X, \mu)$ and $g \in L^\infty(X, \mu) = L^\infty(X, \lambda)$, we have $f' \in L^1(X, \lambda)$, where $f' = fv$, and

$$\lim_\alpha \int_X fT_\alpha g \, d\mu = \lim_\alpha \int_X f'P^*\hat{T}_\alpha Pg \, d\lambda = \lim_\alpha \int_{X_1} Pf'\hat{T}_\alpha Pg \, d\mu$$

$$= \int_{X_1} Pf'\hat{T}Pg \, d\mu = \int_X f'P^*\hat{T}Pg \, d\lambda = \int_X fTg \, d\mu.$$

This completes the proof for $\mathscr{C}$.

To prove the compactness of $\mathscr{C}^+$ it is enough to show that $\mathscr{C}^+$ is a closed subset of $\mathscr{C}$. Suppose that T is a point of closure of $\mathscr{C}^+$ in $\mathscr{C}$, and that $(T_\alpha)_\alpha$ is a net in $\mathscr{C}^+$ converging to T. For each $0 \leqslant f \in L^1(X, \mu)$ and $0 \leqslant g \in L^\infty(X, \mu)$, we have $\langle f, Tg\rangle = \lim_\alpha\langle f, T_\alpha g\rangle \geqslant 0$, so that $T \in \mathscr{C}^+$. The convexity of $\mathscr{C}^+$ is obvious. It follows from Theorem 1 of [10], together with $P\mathscr{C}^+P^* = \mathscr{C}^+(X_1, \mu)$ and $P\Psi_1'P^* = \Psi_1'(X_1, \mu)$, that Ψ_1' is dense in $\mathscr{C}^+$.

We show readily that $\mathscr{C}_0^+$ is a closed subset of $\mathscr{C}^+$. It follows from Theorem 1 of [9], together with $P\mathscr{C}_0^+P^* = \mathscr{C}_0^+(X_1, \mu)$ and $P\Psi_1P^* = \Psi_1(X_1, \mu)$, that Ψ_1 is dense in $\mathscr{C}_0^+$. This completes the proof.

It is easy to see that on the set $\mathscr{C}$ the weak* operator topologies of $[L^\infty(X, \mu)]$ and $[L^\infty(X, \lambda)]$ coincide. Using Theorem 1 and the proof of Theorem 2 in [12] we obtain the following result.

THEOREM 2. *$\mathscr{C}$ [resp. $\mathscr{C}^+$, $\mathscr{C}_0^+$] is the closed convex hull of $\tilde{\Psi}_1$ [resp. Ψ_1', Ψ_1] in the strong operator topology of* $[L^\infty(X, \lambda), L^1(X, \lambda)]$.

THEOREM 3. *$\mathscr{C}$ [resp. $\mathscr{C}^+$, $\mathscr{C}_0^+$] is the closed convex hull of $\tilde{\Psi}$ [resp. Ψ', Ψ] in the strong operator topology of* $[L^\infty(X, \mu)]$.

Proof. Since $\mathscr{C}$ is a closed convex set in the strong operator topology of $[L^\infty(X, \mu)]$ by Theorem 1, it suffices to show that the convex hull of $\tilde{\Psi}$, ch $\tilde{\Psi}$, is dense in $\mathscr{C}$. For each $T \in \mathscr{C}$, let $\hat{T} = PTP^*$ $(\in \mathscr{C}(X_1, \mu))$. By Theorem 3 of [12], there exists a net $(\hat{Q}_\pi)_\pi$ in ch $\tilde{\Psi}(X_1, \mu)$ that converges to $\hat{T}$ in the strong operator topology of $[L^\infty(X_1, \mu)]$. If we define the net $(Q_\pi)_\pi$ by $Q_\pi = P^*\hat{Q}_\pi P$, then $(Q_\pi)_\pi \subset \text{ch}\, \tilde{\Psi}$. We have then for each $g \in L^\infty(X, \mu) = L^\infty(X, \lambda)$,

$$\| Q_\pi g - Tg \|_\infty = \| \hat{Q}_\pi \hat{g} - \hat{T}\hat{g} \|_\infty \to 0,$$

where $\hat{g} = Pg \in L^\infty(X_1, \mu)$. This completes the proof for $\mathscr{C}$.

The above argument, together with [10, Theorem 3] and [9, Theorem 3], proves the theorem for $\mathscr{C}^+$ and $\mathscr{C}_0^+$.

We see easily from Theorem 1 that in the weak* operator topology of $[L^\infty(X, \mu)]$, the sets $\mathscr{M}$, $\mathscr{M}^+$ and $\mathscr{M}_0^+$ are not closed, but they are sequentially closed.

THEOREM 4. *$\mathscr{M}$ [resp. $\mathscr{M}^+$, $\mathscr{M}_0^+$] is the sequential closure of $\tilde{\Psi}_1$ [resp. Ψ_1', Ψ_1] in the weak* operator topology of $[L^\infty(X, \mu)]$.*

Proof. Given $T \in \mathscr{M}$, if we set $\hat{T} = PTP^*$, then $\hat{T} \in \mathscr{M}(X_1, \mu)$. It follows from [12, Theorem 7] that there exists a sequence $(\hat{T}_n)_n$ in $\tilde{\Psi}_1(X_1, \mu)$ that converges to $\hat{T}$ in the weak* operator topology of $[L^\infty(X_1, \mu)]$. If we define the sequence $(T_n)_n$ by $T_n = P^*\hat{T}_n P$, then $(T_n)_n \subset \tilde{\Psi}_1$. Using the proof of Theorem 1 we prove that the sequence $(T_n)_n$ converges to T in the weak* operator topology of $[L^\infty(X, \mu)]$. This completes the proof for $\mathscr{M}$.

Similarly we obtain the desired conclusion for $\mathscr{M}^+$ and $\mathscr{M}_0^+$ by using [10, Theorem 6] and [9, Theorem 4].

THEOREM 5. *$\mathscr{M}$ [resp. $\mathscr{M}^+$, $\mathscr{M}_0^+$] is the sequential closure of ch $\tilde{\Psi}_1$ [resp. ch Ψ_1', ch Ψ_1] in the strong operator topology of $[L^\infty(X, \lambda), L^1(X, \lambda)]$.*

Proof. Let $T \in \mathscr{M}$ and $\hat{T} = PTP^*$ ($\in \mathscr{M}(X_1, \mu)$). By Theorem 7 of [12], there exists a sequence $(\hat{T}_n)$ in ch $\tilde{\Psi}_1(X_1, \mu)$ that converges to $\hat{T}$ in the strong operator topology of $[L^\infty(X_1, \mu), L^1(X_1, \mu)]$. Let $T_n = P^*\hat{T}_n P$ ($n = 1, 2, \ldots$). It follows that $T_n \in \text{ch}\, \tilde{\Psi}_1$ ($n = 1, 2, \ldots$), and that for each $g \in L^\infty(X, \lambda) = L^\infty(X, \mu)$,

$$\int_X |(T_n - T)\, g \mid d\lambda = \int_{X_1} |(\hat{T}_n - \hat{T})\, Pg \mid d\mu \to 0 \qquad \text{as} \quad n \to \infty.$$

This completes the proof for $\mathscr{M}$.

Similar arguments, together with [10, Theorem 6] and [9, Theorem 4] prove the desired conclusions for $\mathscr{M}^+$ and $\mathscr{M}_0^+$.

THEOREM 6. *For each T in $\mathscr{K}$ [resp. $\mathscr{K}^+$, $\mathscr{K}_0^+$], there exists a sequence $(T_n)_n$ in ch $\tilde{\Psi}$ [resp. ch Ψ', ch Ψ] that converges to T in the uniform operator topology of $[L^\infty(X, \lambda), L^1(X, \lambda)]$.*

Proof. Supoose that $T \in \mathscr{K}$ has the kernel $t(x, y)$. Let $\hat{T} = PTP^*$. It follows from Lemma 3 that $T \in \mathscr{K}(X, \lambda)$ has the kernel $t'(x, y) = t(x, y) v(y)$, and that $\hat{T} \in \mathscr{K}(X_1, \mu)$ has the kernel $\hat{t}(x, y) = t' \circ \zeta(x, y)$. By Theorem 8 of [12], there exists a sequence $(\hat{T}_n)_n$ in ch $\tilde{\Psi}(X_1, \mu)$ that converges to $\hat{T}$ in the uniform operator topology of $[L^\infty(X_1, \mu), L^1(X_1, \mu)]$, that is, $\| \hat{T}_n - \hat{T} \|_{\infty,1} \to 0$ $(n \to \infty)$, where

$$\| \hat{T}_n - \hat{T} \|_{\infty,1} = \sup \left\{ \int_{X_1} |(\hat{T}_n - \hat{T})\, \hat{g}|\, d\mu : \| \hat{g} \|_\infty \leqslant 1, \hat{g} \in L^\infty(X_1, \mu) \right\}.$$

If we put $T_n = P^* \hat{T}_n P$, then $T_n \in \text{ch}\, \tilde{\Psi}$ $(n = 1, 2,...)$. It is straightforward to show that $\| T_n - T \|_{\infty,1} = \| \hat{T}_n - \hat{T} \|_{\infty,1}$, where

$$\| T_n - T \|_{\infty,1} = \sup \left\{ \int_X |(T_n - T)\, g|\, d\lambda : \| g \|_\infty \leqslant 1, g \in L^\infty(X, \lambda) \right\},$$

from which the desired conclusion for $\mathscr{K}$ follows.

Similar arguments, together with [10, Theorem 7] and [8, Theorem 3], prove the theorem for $\mathscr{K}^+$ and $\mathscr{K}_0^+$. This completes the proof.

We state the following norm approximation theorem.

THEOREM 7. *If $T \in \mathscr{M}$ [resp. $\mathscr{M}^+$] is a Hilbert-Schmidt operator, then there exists a sequence $(P_k)_k$ in ch $\tilde{\Psi}'$ [resp. ch Ψ'] such that $\| T - P_k \|_2 \to 0$ as $k \to \infty$.*

For each positive integer n, let $D_i^n = [(i-1)\, 2^{-n}, i 2^{-n})$ *and* $e_i^n = (2^n)^{1/2} 1_{D_i^n}$ $(i = 1, 2,...)$. Define the operator U_n by

$$U_n f = \sum_{i=1}^{\infty} \langle f, e_i^n \rangle\, e_i^n \qquad (f \in L^1 \cap L^\infty).$$

It is easy to see that $U_n \in \mathscr{D}_0^+$, $U_n = U_n^*$, and $U_n U_m = U_m U_n = U_n$ if $n \leqslant m$. Note also that U_n is a projection on L^2 and that U_n converges to the identity operator I as $n \to \infty$ in the strong operator topology of $[L^p]$ $(1 \leqslant p < \infty)$.

Define $\mathfrak{H}_n = \{U_n f : f \in L^2\}$. Note that $\mathfrak{H}_n$ is a closed subspace of L^2. For each $T \in [L^2]$, let T_n be the compression of T to $\mathfrak{H}_n$, that is, T_n is the operator on $\mathfrak{H}_n$ defined by the condition $T_n h = U_n T h$ $(h \in \mathfrak{H}_n)$, or equivalently by the condition $T_n U_n f = U_n T U_n f$ $(f \in L^2)$. In terms of the orthonormal basis $(e_i^n)_i$ for $\mathfrak{H}_n$, the compression T_n is uniquely represented by the infinite matrix (t_{ij}^n), where $t_{ij}^n = \langle T e_j^n, e_i^n \rangle$, as follows:

$$T_n(U_n f) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} t_{ij}^n \langle f, e_j^n \rangle\, e_i^n \qquad (f \in L^2).$$

It is easily seen that the adjoint T_n^* of T_n is the compression of T^* to $\mathfrak{H}_n$ and

$$T_n^*(U_n f) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} t_{ji}^n \langle f, e_j^n \rangle e_i^n \qquad (f \in L^2).$$

Note that $\sum_{i=1}^{\infty} (t_{ij})^2 < \infty$ for each j and $\sum_{j=1}^{\infty} (t_{ij})^2 < \infty$ for each i. Call T and T^* dilations (extensions) of T_n and T_n^* to L^2.

Let T be an element of $\mathscr{K}_2$ with kernel $t(x, y)$, and let

$$t_n(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} t_{ij}^n e_i^n(x)\, e_j^n(y) \qquad (n = 1, 2, \ldots).$$

It is easily seen that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} | t_{ij}^n |^2 = \iint_{X \times X} | t_n(x, y)|^2 \, d\mu(x)\, d\mu(y)$$

$$\leqslant \iint_{X \times X} | t(x, y)|^2 \, d\mu(x)\, d\mu(y) < \infty,$$

so that $U_n T U_n$ is an element of $\mathscr{K}_2$ with kernel $t_n(x, y)$. Note also that

$$\| T - U_n T U_n \|_2 \leqslant \| t - t_n \|_{L^2(X \times X)} \to 0 \qquad \text{as} \quad n \to \infty.$$

For each positive integer m, let $\mathfrak{M}(m)$ be the set of $m \times m$-real matrices (s_{ij}) such that $\sum_{j=1}^{m} | s_{ij} | \leqslant 1$ for each i, and let $\mathfrak{R}(m)$ be the set of those matrices (s_{ij}) in $\mathfrak{M}(m)$ such that $| s_{ij} | = 0$ or 1 $(1 \leqslant i, j \leqslant m)$. Define $\mathfrak{M}_0(m) = \{(s_{ij}) \in \mathfrak{M}(m) : \sum_{j=1}^{m} | s_{ij} | = 1$, for each $i\}$ and $\mathfrak{R}_0(m) = \mathfrak{M}_0(m) \cap \mathfrak{R}(m)$. We show readily that each matrix (s_{ij}) in $\mathfrak{M}(m)$ induces the bounded operator $S_n : \mathfrak{H}_n \to \mathfrak{H}_n$ by the condition

$$S_n(U_n f) = \sum_{i=1}^{m} \sum_{j=1}^{m} s_{ij} < f, \quad e_j^n > e_i^n \qquad (f \in L^2).$$

We have also the relation

$$S_n^*(U_n f) = \sum_{i=1}^{m} \sum_{j=1}^{m} s_{ji} < f, \quad e_j^n > e_i^n \qquad (f \in L^2).$$

Note that $\| S_n \|_2 = \| S_n^* \|_2 \leqslant (m)^{1/2}$. Let $\tilde{\Psi}(Y) = \{I_Y T : T \in \tilde{\Psi}\}$, where $Y \subset X$.

Lemma 4. *Let R_n be the operator on $\mathfrak{H}_n$ induced by a matrix (r_{ij}) in $\mathfrak{R}_0(m)$, where $m, n = 1, 2, \ldots$. Then R_n has a dilation $R \in \tilde{\Psi}(Y)$, where $Y = \bigcup_{i=1}^{m} D_i^n$, such that $\| R \|_2 \leqslant (m)^{1/2}$ and $R U_n f = R_n U_n f$ $(f \in L^2)$.*

Proof. Let $J = \{1, 2,..., m\}$. There exist a map $\sigma : J \to J$ and a partition (J_1, J_2) of J such that

$$r_{ij} = \delta_{\sigma(i),j} \ (i \in J_1, j \in J), \ r_{ij} = -\delta_{\sigma(i),j} \ (i \in J_2, j \in J).$$

If we put $E = \bigcup_{i \in J_1} D_i^n$ and $F = \bigcup_{i \in J_2} D_i^n$, then $Y = E \cup F$. Let $\varphi : X \to X$ be a map such that $\varphi : D_i^n \to D_{\sigma(i)}^n$ $(i \in J)$ is an invertible measure-preserving map, and $\varphi : X - Y \to X - Z$, where $Z = \varphi(Y)$, is an invertible measure-preserving map. Note that $\varphi \in \Psi$, and that $\varphi : Y \to Z$ may be neither bijective nor measure preserving. Define $R = (I_E - I_F) T_\varphi$. We see readily that $R \in \tilde{\Psi}(Y) \subset \tilde{\Psi}'$ and $RU_n = R_n U_n$. To prove $\| R \|_2 \leqslant (m)^{1/2}$, it is enough to show $\| R \|_1 \leqslant m$. Then, the Riesz convexity theorem, together with $\| R \|_\infty \leqslant 1$, implies $\| R \|_2 \leqslant (m)^{1/2}$. For each $A \subset Y$ and $j = 1, 2,..., m$, we obtain

$$\| R 1_{A \cap D_j^n} \|_1 = \sum_{i=1}^{m} | r_{ij} | \, \mu(A \cap D_j^n) \leqslant m\mu(A \cap D_j^n),$$

so that $\| R 1_A \|_1 \leqslant m\mu(A)$. Since $\| R 1_A \|_1 = 0$ for each $A \subset X - Y$, we have $\| R \|_1 \leqslant m$. This completes the proof.

Let $\mathfrak{M}^+(m)$ [resp. $\mathfrak{M}_0^+(m)$] be the set of nonnegative matrices in $\mathfrak{M}(m)$ [resp. $\mathfrak{M}_0(m)$]. Note that $\mathfrak{M}^+(m)$ [resp. $\mathfrak{M}_0^+(m)$] consists of $m \times m$-substochastic [resp. stochastic] matrices. Define $\mathfrak{R}^+(m) = \mathfrak{R}(m) \cap \mathfrak{M}^+(m)$ and $\mathfrak{R}_0^+(m) = \mathfrak{R}_0(m) \cap \mathfrak{M}_0^+(m)$. By a minor modification of the proof of Lemma 4 we obtain the following corollary.

COROLLARY. *Let R_n be the positive operator on $\mathfrak{H}_n$ induced by a matrix (r_{ij}) in $\mathfrak{R}^+(m)$ [resp. $\mathfrak{R}_0^+(m)$], where $m, n = 1, 2,....$ Then R_n has a dilation $R \in \Psi(Y')$ [resp. $\Psi(Y)$], where $Y' \subset Y = \bigcup_{i=1}^m D_i^n$, such that $\| R \|_2 \leqslant (m)^{1/2}$ and $RU_n = R_n U_n$.*

In is known [11, Lemma 5; 10, Lemma *E*; 13, p. 133] that

$$\mathfrak{M}(m) = \text{ch}\, \mathfrak{R}_0(m), \ \mathfrak{M}^+(m) = \text{ch}\, \mathfrak{R}^+(m), \ \mathfrak{M}_0^+(m) = \text{ch}\, \mathfrak{R}_0^+(m).$$

Thus we have at once the following result from Lemma 4, together with its corollary.

LEMMA 5. *Let S_n be the operator on $\mathfrak{H}_n$ induced by a matrix (s_{ij}) in $\mathfrak{M}(m)$ [resp. $\mathfrak{M}^+(m)$, $\mathfrak{M}_0^+(m)$], where $m, n = 1, 2,....$ Then S_n has a dilation S in ch $\tilde{\Psi}(Y)$ [resp. ch $\Psi(Y')$, ch $\Psi(Y)$], where $Y' \subset Y = \bigcup_{i=1}^m D_i^n$, such that $\| S \|_2 \leqslant (m)^{1/2}$ and $SU_n = S_n U_n$.*

The following approximation, proved for the case where the underlying space is the closed unit interval [6, Lemma 2.5], may be shown by a minor modification of the argument given in [6, pp. 524, 525].

LEMMA 6. *For each U_n, there exist measure-preserving maps θ_1 and θ_2 from X onto itself such that*

$$\|\{\tfrac{1}{2}(T_{\theta_1} + T_{\theta_2})\}^{2k} - U_n\|_2 \leqslant 2^{-k} \qquad (k = 1, 2,...).$$

Proof of Theorem 7. Suppose that $T \in \mathscr{K}_2$. Let t be the kernel for T and let t_n be the kernel for U_nTU_n ($n = 1, 2,...$). Given $\epsilon > 0$, choose a positive integer n such that $\| t - t_n \|_{L^2(X\times X)} < \epsilon/3$, so that

$$\| T - U_nTU_n \|_2 < \epsilon/3.$$

Choose a positive integer M such that

$$\sum_{i=m+1}^{\infty} \sum_{j=1}^{\infty} | t_{ij}^n |^2 < (\epsilon/3\ \sqrt{2})^2, \qquad \sum_{j=m+1}^{\infty} \sum_{i=1}^{\infty} | t_{ij}^n |^2 < (\epsilon/3\ \sqrt{2})^2,$$

where $m = 2^nM$. Define the matrix (s_{ij}) in $\mathfrak{M}(m)$ by $s_{ij} = t_{ij}^n$, where $1 \leqslant i, j \leqslant m$. Let T_n be the compression of T to $\mathfrak{H}_n$, and let S_n be the operator on $\mathfrak{H}_n$ induced by the matrix (s_{ij}). It is easily seen that

$$\| T_nU_n - S_nU_n \|_2^2 \leqslant \sum_{i=m+1}^{\infty} \sum_{j=1}^{\infty} | t_{ij}^n |^2 + \sum_{j=m+1}^{\infty} \sum_{i=1}^{\infty} | t_{ij}^n |^2 < (\epsilon/3)^2.$$

By Lemma 5, S_n has a dilation S in ch $\tilde{\Psi}[0, M] \subset$ ch $\tilde{\Psi}'$ such that $\| S \|_2 \leqslant (m)^{1/2}$ and $SU_n = S_nU_n$. Choose k_0 such that $2^{-k_0} < \epsilon/3(m)^{1/2}$. By Lemma 6 we can choose $V = \tfrac{1}{2}(T_{\theta_1} + T_{\theta_2})$ with $T_{\theta_i} \in \Phi$ ($i = 1, 2$) such that $\| V^{2k} - U_n \|_2 < \epsilon/3(m)^{1/2}$ ($k > k_0$), so that

$$\| SU_n - SV^{2k} \|_2 < \epsilon/3 \ (k > k_0).$$

Thus we obtain from the inequalities above

$$\| T - SV^{2k} \|_2 < \epsilon \ (k > k_0).$$

If we define $P_k = SV^{2k}$ ($k = 1, 2,...$), then $P_k \in$ ch $\tilde{\Psi}[0, M] \subset$ ch $\tilde{\Psi}'$.

A modification of the above argument, together with Lemma 5, proves the theorem for $T \in \mathscr{M}^+$.

3. APPROXIMATIONS OF $\mathscr{D}$

Following Brown [1] and Peck [14] we call a measure λ on the product space $(X \times X, \mathscr{F} \times \mathscr{F})$ a doubly substochastic (d.s.s.) measure if

$$\lambda(A \times B) \leqslant \min\{\mu(A), \mu(B)\} \ (A, B \in \mathscr{F}).$$

A d.s.s. measure λ is called a doubly stochastic (d.s.) measure if

$$\lambda(A \times X) = \lambda(X \times A) = \mu(A) \; (A \in \mathscr{F}).$$

We see readily that each d.s.s. measure is σ-finite, and that no d.s. measure is finite. The following lemma is a reformulation of Theorems 1 and 2 in [1].

LEMMA 7. (Brown) *There exists a one-to-one correspondence between d.s.s. [resp. d.s.] measures λ and d.s.s. [resp. d.s.] operators T such that*

$$\langle f, Tg \rangle = \iint_{X \times X} f(x)\, g(y)\, d\lambda(x, y) \qquad (f \in L^1, g \in L^\infty). \tag{1}$$

As an immediate consequence of Lemma 7 we obtain the following lemma.

LEMMA 8. *There exists a one-to-one correspondence between finite d.s.s. measures and d.s.s. operators T such that $T : L^\infty \to L^1 \cap L^\infty$.*

Denote by $T \sim \lambda_T$ the correspondence defined in Lemma 7. We shall note that such correspondence does not extend for $\mathscr{D}$. Let $\varphi(x) = x$ on X, and let (A, B) be a partition of X such that $\mu(A) = \mu(B) = \infty$. Define an operator T in $\mathscr{D}$ by $T = (I_A - I_B)\, T_\varphi$. If there were a set function λ on $(X \times X, \mathscr{F} \times \mathscr{F})$ satisfying Eq. (1), then $\lambda(X \times X) = \langle 1, T1 \rangle = \mu(A) - \mu(B) = \infty - \infty$, a contradiction. However, we prove the following Proposition whose statement requires the following definition. By a signed d.s.s. [resp. d.s.] measure λ we shall mean a signed measure λ on $(X \times X, \mathscr{F} \times \mathscr{F})$ such that the total variation $|\lambda|$ of λ is a d.s.s. [resp. d.s.] measure. By definition each signed d.s.s. measure assumed at most one of the values $+\infty$ and $-\infty$. Let λ^+ and λ^- denote, respectively, the upper variation and the lower variation of λ. Note that for each signed d.s.s. measure λ, both λ^+ and λ^- are d.s.s. measures.

PROPOSITION. *There exists a one-to-one correspondence between those operators T in $\mathscr{D}$ [resp. $\mathscr{D}_0$] for which at least one of T^+ and T^- maps L^∞ into $L^1 \cap L^\infty$ and signed d.s.s. [resp. d.s.] measures λ such that*

$$\langle f, Tg \rangle = \iint_{X \times X} f(x)\, g(y)\, d\lambda(x, y) \qquad (f \in L^1, g \in L^\infty). \tag{2}$$

In particular, $\lambda^+ = \lambda_{T^+}$, $\lambda^- = \lambda_{T^-}$, $|\lambda| = \lambda_{|T|}$.

Proof. Let λ be a signed d.s.s. measure with finite λ^-. By Lemmas 7 and 8, there exist d.s.s. operators U, V and W such that $\lambda^+ = \lambda_U$, $\lambda^- = \lambda_V$. $|\lambda| = \lambda_W$ and $V : L^\infty \to L^1 \cap L^\infty$. We see readily that $W = U + V$. If we set

$S = U - V$, then $|S| \leqslant W$, so that $S \in \mathscr{D}$. By a minor modification of the proof of [1, Theorem 2], there exists a unique $T \in \mathfrak{D}$ such that

$$\langle f, Tg \rangle = \iint_{X \times X} f(x)\, g(y)\, d\lambda(x, y) \qquad (f \in L^1,\, g \in L^\infty).$$

We have then $T = S$, so that $T \in \mathscr{D}$. Since $T^+ \leqslant U$ and $T^- \leqslant V$, it follows that $T^- : L^\infty \to L^1 \cap L^\infty$, $|T| \leqslant W$, and that $\lambda_{T^+} \leqslant \lambda^+$, $\lambda_{T^-} \leqslant \lambda^-$, $\lambda_{|T|} \leqslant |\lambda|$. On the other hand, if we set $\lambda_1 = \lambda_{T^+}$ and $\lambda_2 = \lambda_{T^-}$, then $\lambda = \lambda_1 - \lambda_2$, so that $\lambda^+ \leqslant \lambda_1$ and $\lambda^- \leqslant \lambda_2$, or equivalently $U \leqslant T^+$ and $V \leqslant T^-$. Thus, $\lambda^+ = \lambda_1$ and $\lambda^- = \lambda_2$, or equivalently $U = T^+$ and $V = T^-$. We also have $|T| = W$ and $|\lambda| = \lambda_{|T|}$.

Conversely if $T \in \mathscr{D}$ and $T^- : L^\infty \to L^1 \cap L^\infty$, then T^+, T^- and $|T|$ are d.s.s. and by Lemma 7 there exist d.s.s. measures $\lambda_1(= \lambda_{T^+})$, $\lambda_2(=\lambda_{T^-})$ and $\lambda_3(= \lambda_{|T|})$ satisfying Eq. (1), respectively. Note that λ_2 is finite. Define $\lambda = \lambda_1 - \lambda_2$. Then λ is a signed d.s.s. measure with finite λ^- satisfying Eq. (2). This completes the proof for the case of signed d.s.s. measures.

The proof for the case of signed d.s. measures follows from the following fact. For each signed d.s.s. measure $\lambda = \lambda_T$ ($T \in \mathscr{D}$), λ is a signed d.s. measure iff $|\lambda| = \lambda_{|T|}$ is a d.s. measure iff $|T|$ is a d.s. operator, and at least one of T^+ and T^- maps L^∞ into $L^\infty \cap L^1$.

COROLLARY. *There exists a one-to-one correspondence between those T in $\mathscr{D}$ for which both T^+ and T^- map L^∞ into $L^1 \cap L^\infty$ and finite signed d.s.s. measures λ satisfying Eq.* (2).

The proof is immediate from Proposition and Lemma 8.

Let L denote the family of simple functions on X having compact support.

LEMMA 9. *$\mathscr{D}$ is a compact convex set in the weak operator topology of* $[L^2]$.

Proof. The convexity of $\mathscr{D}$ is clear. Since the closed unit ball $\mathscr{C}_2$ of $[L^2]$ is compact in the weak operator topology of $[L^2]$ [3, p. 512], it remains to show that $\mathscr{D}$ is a closed subset of $\mathscr{C}_2$. Let T be a point of closure of $\mathscr{D}$ in $\mathscr{C}_2$. Since the weak operator topology of $[L^2]$ restricted to $\mathscr{C}_2$ is metrizable, there exists a sequence $(T_n)_n$ in $\mathscr{D}$ that converges to T. We have for $f, g \in L$,

$$|\langle f, T^*g \rangle| = |\langle Tf, g \rangle| = \lim_n |\langle T_n f, g \rangle| \leqslant \|f\|_\infty \|g\|_1 \wedge \|f\|_1 \|g\|_\infty. \quad (3)$$

Note that $T^* \in \mathscr{C}_2$. For each $f \in L$ and $g = \operatorname{sgn}(Tf)\, 1_{X_n}$, where $X_n = [0, n]$, we obtain from (3) that $\int_{X_n} |Tf|\, d\mu \leqslant \|f\|_1$. It follows from the monotone convergence theorem that $\|Tf\|_1 \leqslant \|f\|_1$ ($f \in L$). Since L is a dense subset of L^1, T is uniquely extended to an element of $\mathscr{C}_1$, also denoted by T. Similarly we show $T^* \in \mathscr{C}_1$.

We shall show now that T is extended to an element of $\mathscr{C}$. It follows easily from (3) that for each $h : |h| \leqslant 1_{X_n}$, $\int_E |Th| \, d\mu \leqslant \mu(E)$, where E is an arbitrary bounded set, so that $|Th| \leqslant 1$. Thus we have

$$|T| 1_{X_n} = \sup\{|Th| : |h| \leqslant 1_{X_n}\} \leqslant 1 \qquad (n = 1, 2, \ldots),$$

and hence $|T| 1$ is defined by $|T| 1 = \lim_n |T| 1_{X_n}$ $(\leqslant 1)$. It is easily seen that $|T|$ is extended uniquely to an element of $\mathscr{C}^+$. It follows that both T^+ and T^- are extended to elements of $\mathscr{C}^+$, so that T is extended to an element of $\mathscr{C}$. Similar argument leads to $T^* \in \mathscr{C}$.

It is straightforward to show that $\langle Tf, g\rangle = \langle f, T^*g\rangle$ $(f \in L^1, g \in L^\infty)$ from which we conclude $T \in \mathscr{D}$.

Lemma 10. *$\mathfrak{D}$ is a compact convex set in the weak* operator topology of $[L^\infty]$.*

Proof. Since $\mathscr{C}$ is compact in the weak* operator topology of $[L^\infty]$ by Theorem 1, and $\mathfrak{D}$ is convex, it remains to show that $\mathfrak{D}$ is a closed subset of $\mathscr{C}$. Let T be a point of closure of $\mathfrak{D}$ in $\mathscr{C}$, and let $(T_\alpha)_\alpha$ be a net in $\mathfrak{D}$ that converges to T. We have then

$$|\langle f, Tg\rangle| = \lim_\alpha |\langle f, T_\alpha g\rangle| \leqslant \|f\|_1 \|g\|_\infty \wedge \|f\|_\infty \|g\|_1 \ (f, g \in L^1 \cap L^\infty).$$

We show readily from the above relation that $\|Tg\|_1 \leqslant \|g\|_1$ $(g \in L^1 \cap L^\infty)$, so that T is uniquely extended to an element of $\mathscr{C}_1$. Thus $T \in \mathfrak{D}$, and the proof is complete.

Define the L-topology on $\mathscr{C}$ by a subbase of consisting of sets of the form $\{S : |\langle f, (S - T) g\rangle| < \epsilon\}$, where $S, T \in \mathscr{C}$ and $f, g \in L$. It is easily seen that the L-topology is weaker than the weak* operator topology of $[L^\infty]$ on $\mathscr{C}$ and is equivalent to the weak operator topology of $[L^2]$ on $\mathscr{D}$. It follows from Lemma 9 that $\mathscr{D}$ is a closed subset of $\mathscr{C}$ and, hence, is compact in the weak* operator topology of $[L^\infty]$. Note that the weak* operator topology for $[L^\infty]$ is a Hausdorff topology. Since the weak operator topology of $[L^2]$ on $\mathscr{D}$ is a metrizable topology, the weak operator topology of $[L^2]$ and the weak* operator topology of $[L^\infty]$ coincide on $\mathscr{D}$. We shall show that on the set $\mathfrak{D}$, the L-topology is not even a T_1 topology, so that it is strictly weaker than the weak* operator topology of $[L^\infty]$. Let T be a Banach limit on L^∞, $Tf(x) = \mathrm{LIM}_{y \to \infty} f(y)$ $(f \in L^\infty)$, such that $Tf(x) = \lim_{y \to +\infty} f(y)$ if the right hand limit exists [1, p. 370; 3, p. 73]. Then T is a positive linear functional such that $T1 = 1$ and $Tf = 0$ $(f \in L^1)$, so that $T \in \mathfrak{D}$. Since $g_n = 1_{[n,\infty)} \downarrow 0$ as $n \to \infty$, $Tg_n = 1$ $(n = 1, 2, \ldots)$, and $T0 = 0$, we see $T \in \mathfrak{D} - \mathscr{D}$. If we denote the zero operator by 0, then $0 \in \mathscr{D} \subset \mathfrak{D}$ and $0 \neq T$. Since every L-open set containing 0 also contains T, the L-topology is not a T_1 topology on $\mathfrak{D}$.

Following [1] we define the Peck topology on $\mathfrak{D}$ by a subbase consisting of sets of the form

$$\{S : |\langle f, (T - S) g\rangle| < \epsilon, |\langle (T - S) f, g\rangle| < \epsilon\} \ (f \in L^1, g \in L^\infty).$$

By a minor modification of the proof of [1, Theorem 5], we show readily that on $\mathfrak{D}$, the Peck topology is a compact Hausdorff topology and is stronger than the weak* operator topology of $[L^\infty]$, so that by Lemma 10 the two topologies coincide. In particular, the weak operator topology of $[L^2]$, the weak* operator topology of $[L^\infty]$ and the Peck topology coincide on $\mathscr{D}$. It is known [1, p. 370] that $\mathscr{D}^+$ is the closure of Φ_1 in the weak operator topology of $[L^2]$. We prove the following theorem for $\mathscr{D}$.

THEOREM 8. (i) *$\mathscr{D}$ is a compact convex set and is the closure of $\tilde{\Phi}_1$ in the weak operator topology of* $[L^2]$.

(ii) *$\mathscr{D}$ is the closed convex hull of $\tilde{\Phi}_1$ in the strong operator topology of* $[L^2]$.

Proof. (i) In view of Lemma 9, it is enough to show that for each $T \in \mathscr{D}$, there exists $Q \in \tilde{\Phi}_1$ such that

$$|\langle f_i, (T - Q) g_i\rangle| < \epsilon \ (i = 1, 2,..., m),$$

where $f_i, g_i \in L^2$, $\epsilon > 0$, and m is a positive integer. We may assume without loss of generality that f_i and g_i are bounded functions vanishing outside $[0, N]$, where N is a positive integer, and $\|f\|_\infty \leqslant 1$, $\|g\|_\infty \leqslant 1$. Choose a positive integer n such that

$$\| U_n f_i - f_i \|_1 < \epsilon/4, \| U_n g_i - g_i \|_1 < \epsilon/4 \ (1 \leqslant i \leqslant m).$$

Let $Y = [0, N]$ and $S = I_Y T I_Y$. Then $S : L^p(Y) \to L^p(Y)$, $1 \leqslant p \leqslant \infty$, is a contraction operator. It follows from [12, Lemma 10] that there exists an operator $Q' = (I_{Y_1} - I_{Y_2}) T_\varphi$, where (Y_1, Y_2) is a partition of Y, and φ is an invertible measure-preserving map from Y onto Y, such that

$$\langle U_n f, T U_n g\rangle = \langle U_n f, S U_n g\rangle = \langle U_n f, Q' U_n g\rangle \ (f, g \in L^\infty(Y)).$$

Define $\psi \in \Phi_1$ by $\psi = \varphi$ on Y and $\psi(x) = x$ for $x \in X - Y$. If we define $Q = (I_Z - I_{Y_2}) T_\psi$, where $Z = Y_1 \cup (X - Y)$, then $Q \in \tilde{\Phi}_1$ and $\langle U_n f, T U_n g\rangle = \langle U_n f, Q U_n g\rangle$ $(f, g \in L^\infty(Y))$. It follows that for each i $(1 \leqslant i \leqslant m)$,

$$\begin{aligned} |\langle f_i, (T - Q) g_i\rangle| &\leqslant |\langle f_i - U_n f_i, T g_i\rangle| + |\langle U_n f_i, T(g_i - U_n g_i)\rangle| \\ &\quad + |\langle U_n f_i - f_i, Q U_n g_i\rangle| + |\langle f_i, Q(U_n g_i - g_i)\rangle| \\ &\leqslant 2 \|f_i - U_n f_i\|_1 + 2 \|g_i - U_n g_i\|_1 < \epsilon \end{aligned}$$

This proves (i).

Since convex sets have the same closure in both the weak operator and the strong operator topologies for $[L^2]$ [3, p. 477], we obtain (ii). This completes the proof.

It follows from Theorem 8, together with $\tilde{\Phi}_1 \subset \mathscr{D}_0 \subset \mathscr{D}$, that $\mathscr{D}_0$ is not closed in the weak operator topology of $[L^2]$.

For each positive integer m, let $\mathfrak{D}(m)$ denote the set of $m \times m$-real matrices (s_{ij}) such that $\sum_{j=1}^{m} | s_{ij} | \leqslant 1$ for each i, and $\sum_{i=1}^{m} | s_{ij} | \leqslant 1$ for each j. Let $\mathfrak{Q}(m)$ denote the set of those matrices (s_{ij}) in $\mathfrak{D}(m)$ such that $(| s_{ij} |)$ is a permutation matrix. We show readily that the operator $S_n : \mathfrak{H}_n \to \mathfrak{H}_n$ induced by a matrix (s_{ij}) in $\mathfrak{D}(m)$ is a contraction operator, $\| S_n \|_2 \leqslant 1$.

By the strong* operator topology of $[L^2]$ on $\mathscr{D}$ we shall mean the topology induced by a subbase consisting of sets of the form

$$\{S : \|(T - S) f\|_2 < \epsilon, \|(T^* - S^*)\, g\|_2 < \epsilon\} \ (f, g \in L^2).$$

Let $\tilde{\Phi}_1(Y) = \{I_Y T : T \in \tilde{\Phi}_1\}$, where $Y \subset X$, and $\tilde{\Phi}_1' = \{I_Y T : Y \subset X, T \in \tilde{\Phi}_1\}$.

THEOREM 9. *$\mathscr{D}$ is the closed convex hull of $\tilde{\Phi}_1'$ in the strong* operator topology of* $[L^2]$.

LEMMA 11. *Let Q_n be the operator on $\mathfrak{H}_n$ induced by a matrix (q_{ij}) in $\mathfrak{Q}(m)$ where $m, n = 1, 2,\ldots$. Then Q_n has a dilation $Q \in \tilde{\Phi}_1(Y)$, where $Y = \bigcup_{i=1}^{m} D_i^n$, such that*

$$QU_n f = Q_n U_n f \qquad \text{and} \qquad Q^* U_n f = Q_n^* U_n f \qquad (f \in L^2).$$

Proof. Let $J = \{1, 2,\ldots, m\}$. There exists a bijection $\sigma : J \to J$ and a partition (J_1, J_2) of J such that

$$q_{ij} = \delta_{\sigma(i),j} (i \in J_1, j \in J),\ q_{ij} = -\delta_{\sigma(i),j}\ (i \in J_2, j \in J).$$

Let $\varphi \in \Phi_1$ be a map such that $\varphi(D_i^n) = D_{\sigma(i)}^n$ for $i \in J$ and $\varphi(x) = x$ for $x \in D_i^n$, where $i \notin J$. Let

$$E = \bigcup_{i \in J_1} D_i^n, \qquad F = \bigcup_{i \in J_2} D_i^n, \qquad E' = \varphi(E), \qquad F' = \varphi(F).$$

Define $Q = (I_E - I_F)\, T_\varphi$. Then $Y = E \cup F$ and $Q \in \tilde{\Phi}_1(Y) \subset \tilde{\Phi}_1'$. It is easily verified that $I_E T_\varphi = T_\varphi I_{E'}$ and $I_F T_\varphi = T_\varphi I_{F'}$, so that $T_\varphi^* I_E = I_{E'} T_\varphi^*$, $T_\varphi^* I_F = I_{F'} T_\varphi^*$, and $Q^* = (I_{E'} - I_{F'})\, T_\varphi^* \in \tilde{\Phi}_1(Y)$. We show readily that $QU_n = Q_n U_n$ and $Q^* U_n = Q_n^* U_n$. This completes the proof.

The following lemma is an immediate consequence of the above lemma, together with a known result: $\mathfrak{D}(m) = \text{ch}\, \mathfrak{Q}(m)$ [12, Lemma 11].

LEMMA 12. *Let S_n be the operator on $\mathfrak{H}_n$ induced by a matrix (s_{ij}) in $\mathfrak{D}(m)$,* *where $m, n = 1, 2,\ldots$. Then S_n has a dilation $S \in ch\, \tilde{\Phi}_1(Y)$, where $Y = \bigcup_{i=1}^{m} D_i^n$ such that $SU_n = S_nU_n$ and $S^*U_n = S_n^*U_n$.*

Proof of Theorem 9. We see readily from Theorem 8 that $\mathscr{D}$ is closed in the strong* operator topology of $[L^2]$. Thus, it suffices to show that for each T in $\mathscr{D}$, there exists an S in ch $\tilde{\Phi}_1'$ such that $\| Tf_i - Sf_i \|_2 < \epsilon$ and $\| T^*f_i - S^*f_i \|_2 < \epsilon$ $(i = 1, 2,\ldots, k)$, where $f_i, g_i \in L^2$ $(i = 1, 2,\ldots, k)$, k is a positive integer, and $\epsilon > 0$. We may assume without loss of generality that f_i and g_i vanish outside an interval $[0, N]$, where N is a positive integer, and $\| f_i \|_2 \leqslant 1$, $\| g_i \|_2 \leqslant 1$. Choose an n sufficiently large so that

$$\| U_nh - h \|_2 < \epsilon/4, \| Th - U_nTh \|_2 < \epsilon/4, \| T^*h - U_nT^*h \|_2 < \epsilon/4,$$

where $h = f_1,\ldots, f_k, g_1,\ldots, g_k$. Set $m_1 = 2^nN$. Let T_n be the compression of T to $\mathfrak{H}_n$, and let (t_{ij}^n) be the matrix defined by $t_{ij}^n = \langle Te_j^n, e_i^n \rangle$ $(i, j = 1, 2,\ldots)$. Note that $\sum_{i=1}^{\infty} | t_{ij}^n | \leqslant 1$ for each j and $\sum_{j=1}^{\infty} | t_{ij}^n | \leqslant 1$ for each i. Choose a positive integer $m > m_1$ such that

$$\sum_{i=m+1}^{\infty} | t_{ij}^n | < \epsilon/(4(m_1)^{1/2}) \qquad (1 \leqslant j \leqslant m_1),$$

$$\sum_{j=m+1}^{\infty} | t_{ij}^n | < \epsilon/(4(m_1)^{1/2}) \qquad (1 \leqslant i \leqslant m_1).$$

If we define an $m \times m$-matrix (s_{ij}) by $s_{ij} = t_{ij}^n$ $(1 \leqslant i, j \leqslant m)$, then $(s_{ij}) \in \mathfrak{D}(m)$. Let S_n be the operator on $\mathfrak{H}_n$ induced by the matrix (s_{ij}). By Lemma 12, S_n has a dilation S in ch $(\tilde{\Phi}_1')$ such that $SU_n = S_nU_n$ and $S^*U_n = S_n^*U_n$. We have that for each $h = f_1,\ldots, f_k, g_1,\ldots, g_k$,

$$\| T_nU_nh - S_nU_nh \|_2^2 \leqslant \sum_{i=m+1}^{\infty} \sum_{j=1}^{m_1} | t_{ij}^n |^2 \leqslant \sum_{j=1}^{m_1} \left(\sum_{i=m+1}^{\infty} | t_{ij}^n |^2 \right) < (\epsilon/4)^2,$$

$$\| T_n^*U_nh - S_n^*U_nh \|_2^2 \leqslant \sum_{j=m+1}^{\infty} \sum_{i=1}^{m_1} | t_{ij}^n |^2 \leqslant \sum_{i=1}^{m_1} \left(\sum_{j=m+1}^{\infty} | t_{ij}^n |^2 \right) < (\epsilon/4)^2.$$

It follows from the above inequalities that $\| Th - Sh \|_2 < \epsilon$ and $\| T^*h - S^*h \|_2 < \epsilon$ for $h = f_1,\ldots, f_k, g_1,\ldots, g_k$. This completes the proof.

THEOREM 10. *If $T \in \mathscr{D}$ is a Hilbert-Schmidt operator, then there exists a sequence (P_k) in ch $\tilde{\Phi}'$ such that $\| T - P_k \|_2 \to 0$ as $k \to \infty$.*

The proof is immediate from Lemma 12, together with a minor modification of the proof of Theorm 7.

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